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# BUCKLING OF ISOTROPIC AND ORTHOTROPIC CYLINDERS UNDER INDUCED MOMENTS

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**Abstract**—The buckling of isotropic and composite cylinders due to induced moments was investigated. The cylinder wall may be of solid or of sandwich construction. Expressions were derived for the critical buckling moment. Numerical examples for aluminum and glass–epoxy composite cylinders are presented which show that induced moments are unlikely to result in buckling of cylinders under conditions which may arise in practice. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

Numerous investigators have studied the buckling of isotropic and composite cylinders subjected to axial loads, torsional loads, or external pressure. Relatively little attention has been paid to the buckling of cylinders due to induced moments [Dorninger and Rammerstorfer (1990); Kollár (1994)]. Induced moments may arise, for example when there is a temperature difference across the wall of the cylinder (Fig. 1) or during filament winding of composite cylinders where the tension in the fibers creates stresses varying across the wall of the cylinder. The question arises under what conditions, if any, do induced moments cause buckling of the cylinder. This paper is addressed to this problem and specifically examines the buckling of isotropic or orthotropic cylinders under induced moments. Orthotropic cylinders are of interest because the walls of composite cylinders are frequently orthotropic.

## 2. PROBLEM STATEMENT

We consider either an isotropic or an orthotropic cylinder of inner radius $R_i$, wall thickness $h$ and length $L$ (Fig. 2). The wall of the cylinder may be solid or may be of sandwich construction (Fig. 3), the latter consisting of two facesheets and a core (Fig. 3). The cylinder is subjected to circumferential induced moments $M_0$ which are constant in the circumferential direction as well as along the length of the cylinder (Fig. 4). The problem at hand is to determine the magnitude of the induced moment $M_0$ under which the cylinder buckles.

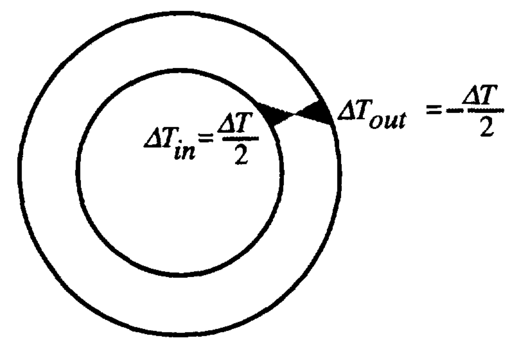


Fig. 1. Temperature distribution across the wall of the cylinder.

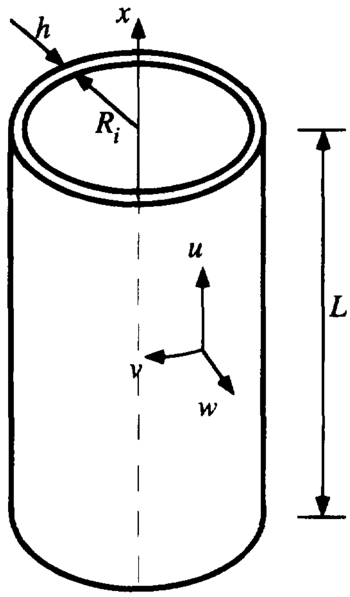

Fig. 2. Geometry of the cylinder.

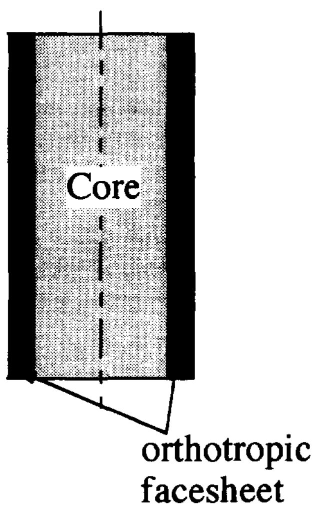

Fig. 3. Build-up of the sandwich wall.

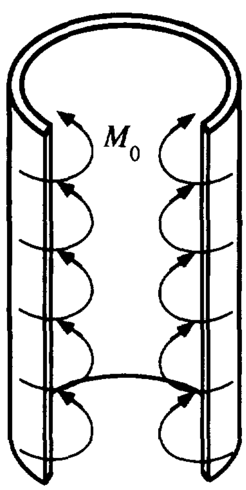

Fig. 4. Induced hoop moment in the wall of the cylinder.

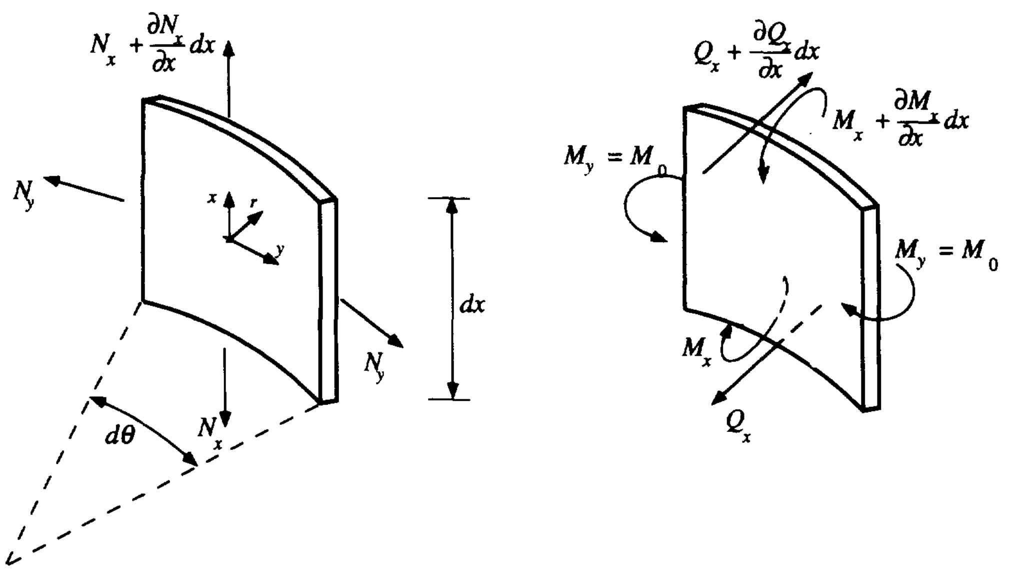


Fig. 5. Internal forces in the wall of the cylinder.

Due to the induced moment $M_0$ the cylinder may buckle into different shapes. Here we consider only axisymmetric buckling along the length of the cylinder.

## 3. ANALYSIS

In the following the analysis is presented for an orthotropic sandwich cylinder (Fig. 2), with deformations of the cylinder wall due to transverse shear taken into account. The results are then reduced to the special cases of solid (non-sandwich) orthotropic cylinders and isotropic cylinders. The starting point of the analysis is the equilibrium equations. The only nonzero forces and moments acting on a segment of the wall are the axial and circumferential in plane forces $N_x$ and $N_y$, the transverse shear $Q_x$ and the moments $M_x$ and $M_0$ (Figs 5a, b).

By neglecting higher-order terms [Timoshenko and Gere (1961), p. 452] radial and axial force balances give (Fig. 5)

$$-Q_x R\,\mathrm{d}\theta + \left(Q_x + \frac{\partial Q_x}{\partial x}\mathrm{d}x\right)R\,\mathrm{d}\theta - N_y\,\mathrm{d}x\,\mathrm{d}\theta = 0, \tag{1}$$

$$N_x\,\mathrm{d}y - \left(N_x + \frac{\partial N_x}{\partial x}\mathrm{d}x\right)\mathrm{d}y = 0. \tag{2}$$

Moment equilibrium about the $y$ axis gives

$$-Q_x R\,\mathrm{d}\theta\,\mathrm{d}x - M_x R\,\mathrm{d}\theta + \left(M_x + \frac{\partial M_x}{\partial x}\mathrm{d}x\right)R\,\mathrm{d}\theta - M_0\,\mathrm{d}x\phi\,\mathrm{d}\theta = 0. \tag{3}$$

In the above equation $R$ is the radius of the center plane of the wall, $\mathrm{d}x$ is the length and $\mathrm{d}\theta$ is the arc of the cylindrical segment under consideration (Fig. 5). $\phi$ is the rotation of the cross-section.

The last term on the left-hand-side of eqn (3) is due to the fact that in the deformed wall the induced moment $M_0$ has a component which creates a moment about the $y$ axis (Figs 6a, b). The magnitude of this moment is of the same order of magnitude as the moments represented by the first, second and third terms in eqn (3). Note that for $\gamma = 0$ (i.e. there is no shear deformation present) this last term is equal to that given by Timoshenko and Gere (1961, p. 453): $M_0\,\mathrm{d}x\partial w/\partial x\,\mathrm{d}\theta$. However, if both bending and shear

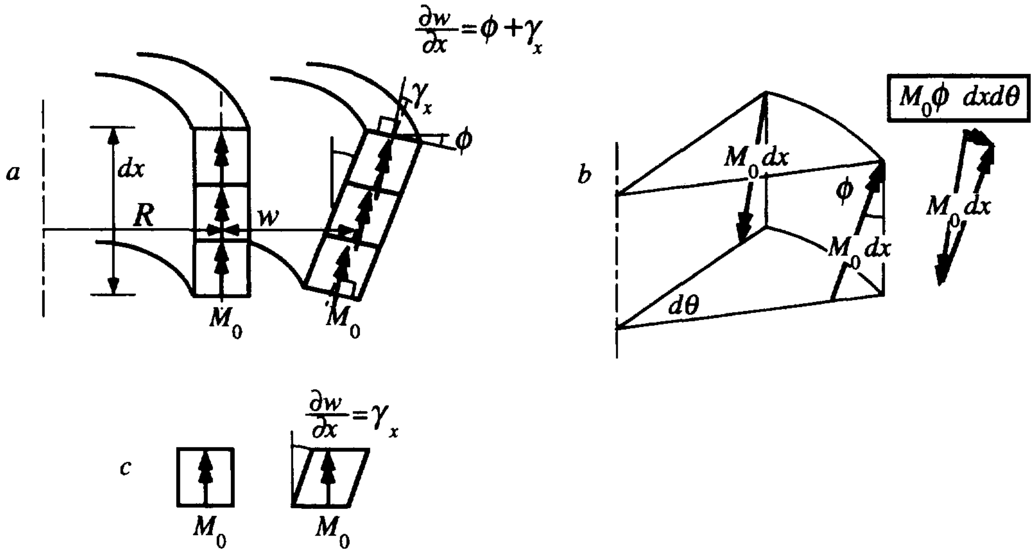


Fig. 6. Bending moment about the $y$ axis due to the change in geometry: (a) and (b) bending and shear deformations, (c) shear deformation only.

deformations are present, $\partial w/\partial x$ should be replaced by the rotation of the cross-section, $\phi$. This is further illustrated in Fig. 6(c), where only shear deformation is present ($\phi = 0, \partial w/\partial x = \gamma$) and the vector of moment, $M_0$, remains vertical.

Equations (1)–(3) may be simplified to yield

$$\frac{1}{R}N_y - \frac{\partial Q_x}{\partial x} = 0 \tag{4}$$

$$\frac{\partial N_x}{\partial x} = 0 \tag{5}$$

$$-Q_x + \frac{\partial M_x}{\partial x} - \frac{1}{R}M_0\phi = 0. \tag{6}$$

The inplane strains $\varepsilon_x$, $\varepsilon_y$, the inplane shear strain $\gamma_{xy}$, the transverse shear strains $\gamma_x$, $\gamma_y$ and the curvatures $\kappa_x$, $\kappa_y$, $\kappa_{xy}$ at the midplane of the cylinder-wall are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{w}{R}, \tag{7}$$

$$\gamma_{xy} = 0, \quad \gamma_x = \frac{\partial w}{\partial w} - \phi, \quad \gamma_y = 0 \tag{8}$$

$$\kappa_x = -\frac{\partial \phi}{\partial x}, \quad \kappa_y = 0, \quad \kappa_{xy} = 0, \tag{9}$$

$u$ and $w$ are axial and radial displacements and $\phi$ is rotation of the cross-section of the wall (Fig. 6).

For an orthotropic cylinder with strains and curvatures given in eqns (7)–(9) the constitutive equations are [Tsai and Hahn (1980)]

$$\begin{aligned} N_x &= A_x\varepsilon_x + A_{xy}\varepsilon_y \\ N_y &= A_{xy}\varepsilon_x + A_y\varepsilon_y \end{aligned} \tag{10}$$

$$M_x = D_x \kappa_x$$
$$Q_x = S_x \gamma_x. \quad (11)$$

where $A$ and $D$ are the inplane and flexural stiffness [Tsai and Hahn (1980), p. 225] and $S_x$ is the shear stiffness [Allen (1969), p. 17]. By combining eqns (4)–(11) and by taking into account that there is no applied axial load (hence $N_x = 0$), we obtain

$$\frac{1}{R^2}\left(A_y - \frac{A_{xy}^2}{A_x}\right)w - S_x\frac{\partial^2 w}{\partial x^2} + S_x\frac{\partial \phi_x}{\partial x} = 0$$
$$-S_x\frac{\partial w}{\partial x} + \left(S_x - \frac{1}{R}M_0\right)\phi - D_x\frac{\partial^2 \phi_x}{\partial x^2} = 0. \quad (12)$$

To proceed with the solution we approximate the radial displacement and the rotation of the cross-section by the trigonometric functions

$$w = \hat{w}\sin(\alpha x), \quad \phi = \hat{\phi}\cos(\alpha x), \quad \text{where} \quad \alpha = \frac{\pi}{l_x}, \quad (13)$$

where $\hat{w}$, $\hat{\phi}$ and $l_x$ are unknown constants. These equations satisfy the conditions of hinged boundaries [Allen (1969)] at $x = 0$ and $x = L$ provided that the ratio $L/l_x$ is a positive integer. Substitution of eqn (13) into eqn (12) yields

$$\begin{bmatrix} \frac{1}{R^2}\left(A_y - \frac{A_{xy}^2}{A_x}\right) + S_x\alpha^2 & -S_x\alpha \\ -S_x\alpha & S_x + D_x\alpha^2 - \frac{1}{R}M_0 \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (14)$$

When the wall buckles $\hat{w}$ and $\hat{\phi}$ are nonzero. Thus, when buckling occurs eqn (14) is only satisfied when the determinant of the coefficient matrix is zero

$$\text{Det}\begin{bmatrix} \frac{1}{R^2}\tilde{A}_y + S_x\alpha^2 & -S_x\alpha \\ -S_x\alpha & S_x + D_x\alpha^2 - \frac{1}{R}M_0 \end{bmatrix} = 0, \quad (15)$$

where

$$\tilde{A}_y = A_y\left(1 - \frac{A_{xy}^2}{A_x A_y}\right). \quad (16)$$

Solution of eqn (15) yields

$$M_0^{cr} = \left(\frac{R\alpha^2}{\tilde{A}_y} + \frac{1}{S_x R}\right)^{-1} + D_x R\alpha^2. \quad (17)$$

which depends on the buckling length. We are interested in the smallest value of the induced moment, $M_0$, which causes buckling (critical buckling moment). Assuming that the buckling length, $l_x$, is much smaller than the length of the cylinder, $L$, the critical induced buckling moment is achieved when the following condition is met

$$\frac{\partial M_0^{cr}}{\partial R\alpha^2} = 0, \quad \text{i.e.} \quad \frac{\partial}{\partial R\alpha^2}\left\{\left(\frac{R\alpha^2}{\tilde{A}_y} + \frac{1}{S_x R}\right)^{-1} + D_x R\alpha^2\right\} = 0. \tag{18}$$

Equation (18) gives

$$R\alpha^2 = \tilde{A}_y\left(\frac{1}{\sqrt{\tilde{A}_y D_x}} - \frac{1}{S_x R}\right). \tag{19}$$

$R\alpha^2$ must be positive. When $R\alpha^2$ is positive $S_x R$ must be greater than $\sqrt{\tilde{A}_y D_x}$ ($S_x R > \sqrt{\tilde{A}_y D_x}$) and the critical induced buckling moment becomes eqns (17) and (19)

$$M_0^{cr} = 2\sqrt{\tilde{A}_y D_x}\left(1 - \frac{\sqrt{\tilde{A}_y D_x}}{2S_x R}\right), \quad \text{if} \quad S_x R > \sqrt{\tilde{A}_y D_x}. \tag{20}$$

When $S_x R$ is equal or less than $\sqrt{\tilde{A}_y D_x}$ then $R\alpha^2$ (which cannot be negative) is set equal to zero to obtain the lowest value of the critical induced moment, $M_0^{cr}$. The critical induced moment becomes (eqn (17))

$$M_0^{cr} = S_x R, \quad \text{if} \quad S_x R \leqslant \sqrt{\tilde{A}_y D_x}. \tag{21}$$

The critical induced moment given by eqn (21) represents pure shear buckling.

*Solid wall cylinders*

For a cylinder with solid wall the shear deformation may be neglected ($S_x = \infty$) and the critical induced moment becomes (eqn (20))

$$M_0^{cr} = 2\sqrt{\tilde{A}_y D_x} \quad \text{(orthotropic)}. \tag{22}$$

For an isotropic material we have [Timoshenko and Gere (1969)]

$$D_x = D_y = \frac{Eh^3}{12(1-\nu^2)}, \quad A_x = A_y = \frac{Eh}{1-\nu^2} \quad A_{xy} = \nu A_y \tag{23}$$

and the calculated critical induced moment becomes

$$M_0^{cr} = \frac{Eh^2}{\sqrt{3(1-\nu^2)}} \quad \text{(isotropic)}. \tag{24}$$

## 4. DISCUSSION

The analysis presented in this paper shows that, in principal, induced moments may cause buckling of cylinders made of an isotropic or orthotropic material. The question remains whether or not in practice such induced moments are sufficiently large so as to result in buckling. To address this question, numerical examples were worked out. In the examples aluminum and glass–epoxy cylinders are considered. The walls of the aluminum cylinders are "solid", while the walls of the glass–epoxy cylinders are of either solid or sandwich construction. The lay-up of the "solid" glass–epoxy composite wall is either $[\pm\phi_{20}/0_{10}]_s$ or $[\pm\phi_{20}/90_{10}]_s$ with the 0° direction being parallel to the cylinder axis (Fig. 2). The lay-up of the facesheet of the sandwich construction is $[\pm 85_{10}]$. The material properties

Table 1. Material properties for aluminum, for glass–epoxy and for the aluminum honeycomb core

| | |
|---|---|
| **Material properties of aluminum** | |
| Young's modulus | $E = 73.0$ GPa, |
| Poisson's ratios | $\nu = 0.33$, |
| Thermal expansion coefficient | $\alpha = 23.9 \times 10^{-6}$°C. |
| **Material properties of glass–epoxy** | |
| Stiffness parameters | |
| longitudinal Young's modulus | $E_x = 38.6$ GPa, |
| transverse in-plane Young's modulus | $E_y = 8.27$ GPa, |
| shear moduli | $G_{xy} = 4.14$ MPa, |
| Poisson's ratios | $\nu_{yx} = 0.26$. |
| Thermal expansion coefficients | |
| longitudinal | $\alpha_x = 8.6 \times 10^{-6}$°C, |
| transverse | $\alpha_y = 22.1 \times 10^{-6}$°C. |
| **Material properties of the aluminum honeycomb core** | |
| Shear moduli $G_{xz} = G_{yz} = 138$ MPa | $G_{xy} \approx 0$, |
| In-plane Young's modulus | $E \approx 0$. |

used in calculations for aluminum, glass–epoxy and the aluminum honeycomb core are given in Table 1.

It is assumed that there is a $\Delta T$ temperature difference across the wall of the cylinder (Fig. 1). Such temperature difference may arise, for example, if the cylinder contains a fluid which is warmer than the ambient. We make the approximation that the temperature varies linearly across the wall. In this case, for an isotropic or orthotropic cylinder the induced moment is (Appendix)

$$M_0 = \frac{\Delta T}{h}(\alpha_x D_{xy} + \alpha_y D_y) \tag{25}$$

where $\alpha_x$ and $\alpha_y$ are the coefficients of thermal expansion in the wall of the cylinder in the axial and circumferential directions, respectively. By substituting eqn (25) into eqns (24), (22) and (20)–(21) we obtain the following expressions for the critical temperature difference $\Delta T_{\text{crit}}$ at which the cylinder buckles†

isotropic solid wall

$$\Delta T_{\text{crit}} = \frac{4\sqrt{3}}{\alpha}\sqrt{\frac{1-\nu}{1+\nu}}, \tag{26}$$

orthotropic, solid wall

$$\Delta T_{\text{crit}} = \frac{2h\sqrt{\tilde{A}_y D_x}}{\alpha_x D_{xy} + \alpha_y D_y}, \tag{27}$$

orthotropic, sandwich wall

$$\Delta T_{\text{crit}} = \frac{2h\sqrt{\tilde{A}_y D_x}\left(1 - \dfrac{\sqrt{\tilde{A}_y D_x}}{2S_x R}\right)}{\alpha_x D_{xy} + \alpha_y D_y} \quad \text{if} \quad S_x R > \sqrt{\tilde{A}_y D_x} \tag{28a}$$

$$\Delta T_{\text{crit}} = \frac{hS_x R}{\alpha_x D_{xy} + \alpha_y D_y} \quad \text{if} \quad S_x R \leqslant \sqrt{\tilde{A}_y D_x} \tag{28b}$$

The calculated critical temperature difference for a solid aluminum cylinder is (eqn (26))

† Note that temperature difference $\Delta T$ induces bending moments not only in the hoop but also in the axial direction. However, in the case of axisymmetric buckling shape, the axial bending moment does not influence eqns (1)–(3) and eqns (20)–(24).

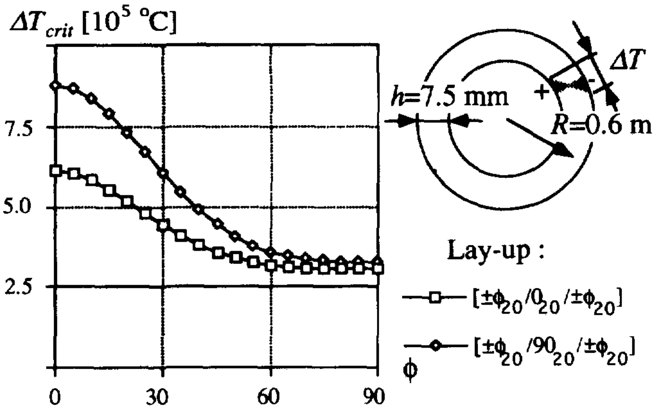


Fig. 7. Critical temperature differences of glass–epoxy cylinders as a function of the winding angle, $\phi$. The transverse shear deformations were neglected.

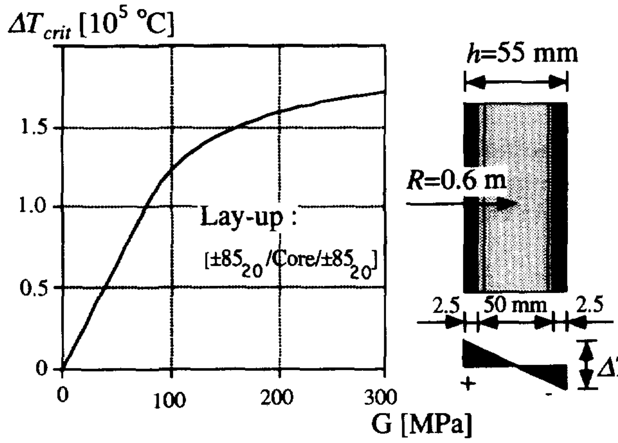


Fig. 8. Critical temperature differences of glass–epoxy sandwich cylinders as a function of the shear modulus of the core. The realistic values of the shear modulus are above 100 MPa. The shear stiffness of the sandwich, $S_x$ was calculated as follows: $S_x = G \times 52.5$ mm.

$\Delta T_{crit} = 2.058 \times 10^{5}$°C, the critical temperature differences for glass–epoxy solid and sandwich cylinders are shown in Figs 7 and 8. As can be seen, unrealistic large temperature differences (on the range of $10^{5}$°C) would be needed to introduce buckling. Thus, it appears that induced moments may cause bucking only in principal. Induced moments are most unlikely to lead to bucking of cylinders made of "practical" materials.

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## REFERENCES


Allen, H. G. (1969) *Analysis and Design of Structural Sandwich Panels*. Pergamon Press, Oxford.

Dorninger, K. and Rammerstorfer, F. G. (1990) A layered composite shell element for elastic and thermoelastic stress and stability analysis at large deformations. *International Journal of Numerical Methods in Engineering* **30**, 833–858.

Kollár, L. P. (1994) Buckling of anisotropic cylinders. *Journal of Reinforced Plastics and Composites*. **13–11**, 954–975.

Timoshenko, S. P. and Gere, J. M. (1961) *Theory of Elastic Stability*, 2nd edn. McGraw-Hill, New York.

Tsai, S. W. and Hahn, H. T. (1980) *Introduction to Composite Materials*. Technomic, Lancaster, Basel.


## APPENDIX

### *Induced moments by linear change in temperature*

An orthotropic shell is subjected to a change in temperature which varies linearly through the thickness (Fig. 1). The change in temperature on the innermost and outermost surfaces are denoted by $\pm\Delta T/2$. The goal is to determine the induced bending moments in the shell (due to the change in temperature) assuming that the curvatures remain unchanged.

The Cartesian coordinates $x$ and $y$ coincide with the axes of orthotropy. The temperature strains in the shell are as follows

$$\varepsilon_x^T = -z\alpha_x\frac{\Delta T}{h}, \quad \varepsilon_y^T = -z\alpha_y\frac{\Delta T}{h} \tag{A1}$$

where $\alpha_x$ and $\alpha_y$ are the coefficients of thermal expansion in the $x$ and $y$ direction, respectively, $z$ is the distance measured from the middle surface of the shell and $h$ is the thickness of the shell. The linearly varying strains, $\varepsilon_x^T$ and $\varepsilon_y^T$ represent the following changes in curvatures.

$$\kappa_x^T = -\alpha_x\frac{\Delta T}{h}, \quad \kappa_y^T = -\alpha_y\frac{\Delta T}{h}. \tag{A2}$$

The total curvature is zero, hence the temperature strains (curvatures) are equal to the opposite of the mechanical strains (curvatures). The induced moments are [Tsai and Hahn (1980), p. 225]

$$M_x = -(\kappa_x^T D_x + \kappa_y^T D_{xy}), \quad M_y = -(\kappa_x^T D_{xy} + \kappa_y^T D_y). \tag{A3}$$

Introducing eqn (A2) into eqn (A3), we obtain

$$M_x = \frac{\Delta T}{h}(\alpha_x D_x + \alpha_y D_{xy}), \quad M_y = \frac{\Delta T}{h}(\alpha_x D_{xy} + \alpha_y D_y). \tag{A4}$$

Equation (A4) is valid for a homogeneous, orthotropic material. For a layered structure the effective coefficient of thermal expansion must be determined from the laminated plate theory [Tsai and Hahn (1980), p. 329].